Around Bohr's thesis

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Bohr's thesis [2] deals with ordinary Dirichlet series

$$\sum_1^\infty a_n n^{-s}. \tag{1}$$

I shall try to describe the mathematical context at the time when he wrote the thesis, 1909, then – very shortly – his main results (from 1909 to 1951) and the problems he left open, then the role of series

$$\sum_1^\infty \mp n^{-s} \tag{2}$$

and the present state of Bohr's problems. Short proofs of classical things are given at the end.

1. *Before 1909*

One century ago, three days after Harald Bohr was born, J. L. W. V. Jensen, a telephone engineer from Copenhagen, presented a paper at the French Academy of Sciences, entitled "Sur la fonction $\zeta(s)$ de Riemann" [14]. H. Bohr quotes it in his first paper, in his thesis, and a number of times: obviously it has been a source of inspiration for him. When he evokes Jensen, he says he was "one of the most gifted mathematicians our country has ever produced".

Jensen was interested i Dirichlet series. He introduced the basis formula

$$|e^{-\lambda_n s} - e^{-\lambda_{n+1} s}| \le \frac{|s|}{|\sigma|}(e^{-\lambda_n \sigma} - e^{-\lambda_{n+1}\sigma})$$

which allowed him to prove that, if a Dirichlet series $\Sigma\, a_n e^{-\lambda_n s}$ converges at a point (say, 0), it converges uniformly on every compact set which lies strictly at the right [13]. His paper on $\zeta(s)$ was motivated by two reasons: first, give a simple proof, not using the functional equation of Riemann, that $(1-s)\zeta(s)$ is an entire function; then, taking for granted what Stieltjes had claimed two years before [27] – namely that he had a proof of the Riemann hypothesis – investigate the location of the first zeros on the critical line.

Stieltjes was also interested in Dirichlet series. In order to derive from the Riemann hypothesis – which he thought he had proved – results on prime numbers he needed multiplication of Dirichlet series. And he stated a curious result: that the product of two Dirichlet series

$$\Sigma\, a_n n^{-s} \quad , \quad \Sigma\, b_n n^{-s},$$

(namely $\Sigma\, c_n n^{-s}$ with $c_n = \sum_{mp=n} a_n b_p$) converges for $\sigma > \frac{1}{2}$ if the two first converge for $\sigma > 1$ [28].

In the last decade of the 19th century, Dirichlet series was an interesting topic. The main work was Cahen's thesis, in France, with a number of formulas for the coefficients, the abscissa of convergence, etc. [8]. Also, with a wrong statement, namely that the theorem of Stieltjes on multiplication of Dirichlet series could be improved, replacing $\sigma > \frac{1}{2}$ by $\sigma > 0$ in the conclusion. Hadamard [9] and de la Vallée Poussin [29] proved that $\zeta(s)$ has no zero on $\sigma = 1$ and derived the prime number theorem.

However, around 1900, there was a decline of interest for Dirichlet series, together with a renewal of interest for *Fourier series*, mainly because of the Lebesgue integral and the Fejér summation theorem.

Then, suddenly, at the time Harald Bohr began to work, a number of first class mathematicians turned again to Dirichlet series. In 1907 and 1908, there were several papers of Landau [17], [18], a short article by Hadamard [10], an extensive study by O. Perron [20], and the important thesis of Schnee [25]. Landau published the first proof of Stieltjes's statement and observed that Cahen had been wrong on multiplication of Dirichlet series. Schnee, among other results, proved that a Dirichlet series converges for Re $s > \sigma_0$ whenever the function $f(s) = \sum_0^\infty a_n e^{-\lambda_n s}$ exists for Re s large and can be extended as a function of order 0 in Re $s > \sigma_0$, that is

$$f(\sigma + it) = O(|t|^\varepsilon) \quad (\sigma > \sigma_0,\ |t| \to \infty)$$

for each $\epsilon > 0$ [25] [26].

In 1908 again, Lindelöf proved his famous convexity theorem about the order of a function. With Bohr's notations, if we write

$$\mu(\sigma) = \inf\{a \mid f(\sigma + it) = O(|t|^a)\} \quad (t \to \infty)$$

when $f(s)$ is holomorphic in the strip $a < \sigma < b$ $(s = \sigma + it)$, then $\mu(\sigma)$ is a convex function [19].

In 1909, Marcel Riesz published three important notes in Comptes-Rendus, all of them on Dirichlet series [22], [23], [24].

2. *Bohr's results and problems (1909–1950)*

1909 is the year Harald Bohr writes his thesis. It begins with a note aux Comptes-Rendus, his first paper, *11th of January*, 1909, "Sur la série de Dirichlet" [1]. And the year ends with the approval of the thesis, signed by the dean on *December 31, 1909*. In between, he writes also a paper for the Göttinger Nachrichten, on the summability of Dirichlet series, the topic of his first note [3]. His starting point is like this: the series $\Sigma\ (-1)^n n^{-s}$, which represents $\zeta(s)\ (1-2^{1-s})$, is summable by the Cesàro process of order r when $\sigma \geq -r$, therefore represents an entire function (a still shorter proof than Jensen's).

Actually, after 1909, his main interest shifted to the ζ–function, then, after 1920, to the theory of almost periodic functions. Nevertheless, his last papers, around 1950, all deal with the problems he considered in his thesis [4], [5], [6], [7].

I shall review at the same time what he did in his thesis and the improvements he gave in the 1950's.

A: The convergence problem

Let σ_a be the abscissa of absolute convergence and σ_c the abscissa of convergence of an ordinary Dirichlet series. Then

$$\sigma_c \leqslant \sigma_a \leqslant \sigma_c + 1 \text{ and } \mu(\sigma_a) = 0 \text{ (obvious)}$$

$$\mu(\sigma_c) \leqslant 1 \text{ (Jensen)}$$

$$\mu(\sigma) = 0 \Rightarrow \sigma_c \leqslant \sigma \text{ (Schnee)}$$

Is it possible to improve, that is, to obtain more information on σ_c from the order function $\mu(.)$ or more information on $\mu(.)$ from the abscissa of convergence σ_c? The answer is negative and it is provided by two examples: a lacunary series of the form

$$\sum_1^\infty (p_n^{-s} - (p_n + 1)^{-s}) \tag{3}$$

gives figure 1, and a more complicated example figure 2.

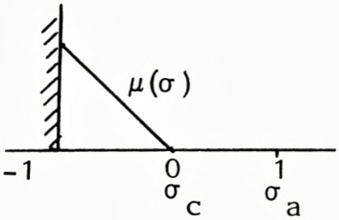

Fig. 1.

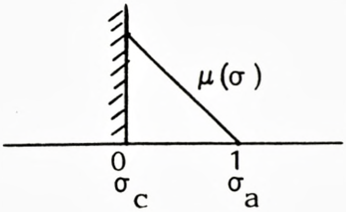

Fig. 2

Figure 2 answers the Stieltjes-Cahen-Landau multiplication problems, because

$$\mu(\sigma\,;f^2) = 2\mu(\sigma\,;f)$$

therefore, with f as in figure 2,

$$\sigma_c(f^2) \geq \frac{1}{2}$$

by Jensen. Bohr was very happy of this discovery and came back to the multiplication problem later [5] [6]. Let me remark that

$$\sigma_c(f^k) \geq 1 - \frac{1}{k}, \quad k = 2, 3, \dots ,$$

a converse of an extended Stieltjes theorem (see appendix).

The conclusion of Bohr is that σ_c is not very well connected with intrinsic properties of $f(s)$, at least not with the order function $\mu(\sigma)$. Henry Helson reconsidered the question in 1962 and gave a very elegant fomula for σ_c using Fourier properties of $f(s)/s$ considered as a function of t $(s = \sigma + it)$ [11].

B: The summability theory

Given an ordinary Dirichlet series (1), let us write now

$$\lambda_0 = \sigma_c$$

$$\lambda_r = \text{abscissa of } C^r\text{-summability of (1)}$$

where C^r is the Cesàro process of summation of order r. In 1909, Bohr considers only integral values of r; in the 1950's, following M. Riesz, general $r > 0$. The "summability function" is $\psi(\sigma)$ defined by

$$\psi(\lambda_r) = r,$$

that is, (1) is C^r-summable at $s = \sigma + it$ if $r < \psi(\sigma)$ and is not C^r-summable at any $s = \sigma + it$ such that $r > \psi(\sigma)$. Bohr's theory leads to

$$\psi(\sigma) \leqslant \mu(\sigma) \leqslant \psi(\sigma) + 1 \tag{4}$$

together with

$$\begin{cases} \psi \text{ convex and } \psi(\sigma) = 0 \text{ for large } \sigma \\ \psi'(\sigma - 0) \leqslant -1 \text{ or else } \psi(\sigma) = 0 \end{cases} \tag{5}$$

(figure 3). As a consequence, the half plane where (1) is C^r-summable for some $r > 0$ and the maximal half plane where the function $f(s)$ represented by (1) is holomorphic and of bounded order are the same (up to the boundary), a striking and final result – actually, the best result of his thesis –.

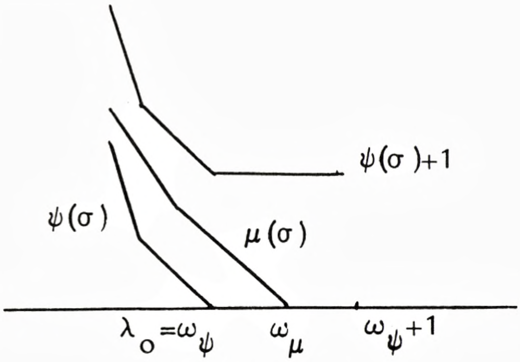

Fig. 3

However, Bohr was not satisfied. Given two functions $\psi(.)$ and $\mu(.)$ as in figure 3, is it possible to construct an ordinary Dirichlet series having them as summability and order function respectively? In his last paper [7] Bohr solves the question completely as far as $\psi(.)$ is concerned: (5) is necessary and sufficient for $\psi(.)$ to be a summability function. What about $\mu(.)$? Assuming (4) and the analogue of (5) for $\mu(.)$, that is $\mu(.)$ is convex and

$$\mu'(\omega_\mu - 0) \leqslant -1 \tag{6}$$

(where $\omega_\mu = \inf\{\sigma;\ \mu(\sigma) = 0\}$), then Bohr shows that $\{\psi(.), \mu(.)\}$ is actually a couple {summability function, order function}.

Now, is (6) a necessary condition (when the first member exists)? This is the last problem of Harald Bohr [7].

Here is a previous problem [4]. Does there exist a Dirichlet series (1) with $\sigma_c = 0$, $\sigma_a = 1$, $\mu(\sigma) = \sup(0, \frac{1}{2} - \sigma)$ (figure 4)?

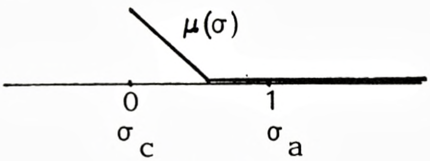

Fig. 4

Both questions are inspired by the Riemann ζ-function. If (6) were a necessary condition, it would prove the Lindelöf hypothesis for $\zeta(s)$, that is

$$\zeta(\frac{1}{2} + it) = O(|t|^{\epsilon}) \ (t \to \infty)$$

for all $\epsilon > 0$. If the Lindelöf hypothesis is true, $\sum_1^\infty (-1)^n n^{-s}$ provides a positive answer to the second question.

3. *After 1951 (a personal selection)*

I already mentioned Helson's formula for σ_c. Following the same idea – Fourier methods in Dirichlet series – Helson gave a very elegant proof of the prime number theorem [12].

Playing with $\mp$ in series (2) gives interesting problems and results. I introduced the game in 1974 and it was developed by H. Queffelec [15] [21]. The first interesting example is

$$\sum_1^\infty \epsilon_n ((2n-1)^s - (2n)^s) \tag{7}$$

with $\epsilon = (\epsilon_1, \epsilon_2, ...) \in \{-1, 1\}^\infty = \Omega$. If we consider Ω as a probability space with the natural probability, figure 4 holds almost surely, which solves the second-mentioned Bohr problem. If we consider Ω as a topological space, then figure 2 holds quasi-surely (meaning: on a dense G_δ-set), which replaces a rather technical construction in Bohr's thesis.

Instead of differences of the first order in (7) it is possible to consider differences of higher and higher order, and get Dirichlet series for which $\mu(\sigma) = \sup\,(0, \frac{1}{2} - \sigma)$ on $(-\infty,\infty)$ (almost surely) or $\mu(\sigma) = \sup(0, 1-\sigma)$ on $(-\infty,\infty)$ (quasi-surely). That helps in constructing the "building blocks" from which Bohr's theorem on $\{\psi(.), \mu(.)\}$ derives (see [21] and [16]).

Quite different results are obtained by Queffelec [21] for almost sure and quasi sure properties of Euler products

$$\prod_1^\infty (1 + \epsilon_n n^{-s})$$

General random Dirichlet series

$$\sum_1^\infty a_n(\omega)\, e^{-\lambda_n s}$$

and their growth properties are considered by Yu Jia-rong [30].

The Harald Bohr centenary was a good opportunity to investigate the last problem, on the order functions $\mu(\sigma)$ of ordinary Dirichlet series. Here are the results I obtained:

1) a necessary condition is $\mu(\sigma + \mu(\sigma) + \frac{1}{2}) = 0$.

2) (6) is not necessary. It is possible to have $\mu'(\omega_\mu - 0)$ as near $\frac{1}{2}$ as one wants [16].

Therefore, the last problem splits into two parts:

1) find another approach to the Lindelöf hypothesis,

2) characterize the order functions of ordinary Dirichlet series. For example, for which $\alpha \geq 0$ can one have

$$\mu(\sigma) = \sup(0, \frac{1}{2} - \sigma, \alpha(1-\sigma))?$$

($\alpha \leqslant \frac{1}{2}$ is necessary, $\alpha = 0$ is sufficient).

4. *Appendix*

1. Here is the proof of Stieltjes's theorem on mulitplication of Dirichlet series. We consider $\Sigma\, a_n n^{-s}$, $\Sigma\, b_n n^{-s}$ and their product $\Sigma\, c_n n^{-s}$. Assume that $\Sigma\, a_n$ and $\Sigma\, b_n$ converge. Given N,

$$\sum_1^N c_n = \sum_{(m,p):mp\leqslant N} a_m b_p$$

$$= \sum_{1\leqslant m\leqslant \sqrt{N}} \Big(a_m \sum_{1\leqslant p\leqslant N/m} b_p\Big) + \sum_{1\leqslant p\leqslant \sqrt{N}} \Big(b_p \sum_{\sqrt{N}<m\leqslant N/p} a_m\Big)$$

$$= O(\sqrt{N}),$$

hence $\Sigma\, c_n n^{-\sigma}$ converges for $\sigma > \frac{1}{2}$, QED.

In the same way, given k series $\Sigma\, a_n^{(j)} n^{-s}$ $(j = 1, 2, \ldots k)$ which converge for $s = 0$, their product is a Dirichlet series which converges for $\sigma > 1 - \frac{1}{k}$.

2. I mentioned the beautiful arguments of Jensen and Bohr proving that $(1-s)\zeta(s)$ is an entire function. However the classical proof is the Rieman functional equation. Here is a simple way to express the proof of the functional equation. Let $E(x)$ = integral part of x for $x \geq 0$, $E(-x) = E(x)$. Then

$$\zeta(s) = \int_0^\infty x^{-s}\, d(E(x) - x)$$

for $0 < \mathrm{Re}\, s < 1$ and

$$d(E(x) - x) = \int_0^\infty (e^{2\pi i t x} + e^{-2\pi i t x})\, d(E(t) - t)$$

in the sense of Schwartz. Through a simple regularisation (multiplying x^{-s} by a C^{∞}-function with compact support in $]0, \infty[$) we have

$$\zeta(s) = \int_0^{\infty} \left(\int_0^{\infty} x^{-s} \left(e^{2\pi itx} + e^{-2\pi itx}\right) dx \right) d(E(t)-t)$$
$$= C(s) \int_0^{\infty} t^{s-1}\, d(E(t)-t) = C(s)\, \zeta(1-s)$$

with $C(s) = 2 \int_0^{\infty} x^{-s} \cos 2\pi x \, dx$.

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